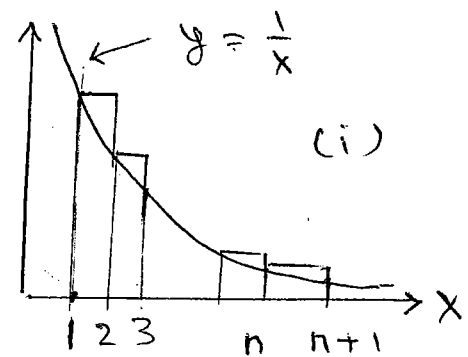
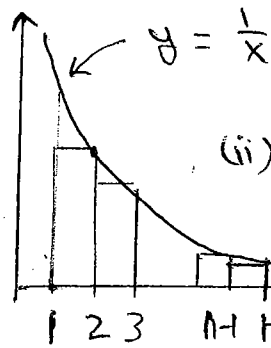


CALCULUS II TUTORIALS (4): SOLUTIONS

① Consider $y = f(x) = \frac{1}{x}$

Then $\sum_{i=1}^{\infty} \frac{1}{n} = \sum_{i=1}^{\infty} f(n)$.

Compare area of rectangles to area under curve.



(i) $\Rightarrow 1 + \frac{1}{2} + \dots + \frac{1}{n} \geq \int_1^{n+1} \frac{dx}{x} = \ln(n+1) - \ln 1 = \ln(n+1)$

(ii) $\Rightarrow \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \leq \int_1^n \frac{dx}{x} = \ln n \Rightarrow 1 + \frac{1}{2} + \dots + \frac{1}{n} \leq 1 + \ln n$

Since $\ln n \rightarrow \infty$ as $n \rightarrow \infty$, (i) $\Rightarrow \sum_{i=1}^{\infty} \frac{1}{n}$ diverges.

Number of seconds per year = $365 \times 24 \times 60 \times 60 \approx 3 \times 10^7$

Age of Universe = 13×10^9 years $\approx (13 \times 10^9) \times (3 \times 10^7) \approx 4 \times 10^{17}$

$\ln(4 \times 10^{17}) \approx 2.3 \log_{10}(4 \times 10^{17}) = 2.3 \times 17.6 \approx 40$

$\Rightarrow S_n$ has only reached 40 by present time!

Subtracting $\ln n$ from (i) and (ii) gives

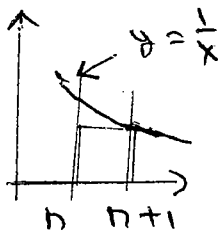
$0 < \ln(n+1) - \ln n \leq 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \leq 1$

$\Rightarrow a_n$ has a lower bound. Also comparing areas

$\frac{1}{n+1} < \int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n \Rightarrow \frac{1}{n+1} - \ln(n+1) + \ln n < 0$

Since a_n is decreasing sequence, there is a greatest upper bound

$\Rightarrow 1 + \frac{1}{2} + \dots + \frac{1}{n} - \ln n \rightarrow \gamma \approx 0.5772$ (Euler's constant)



② We apply integral test using $y = \frac{1}{x(\ln x)^p}$. Then

$\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \int_2^{\infty} \frac{d(\ln x)}{(\ln x)^p} = \frac{1}{1-p} [(\ln x)^{1-p}]_2^{\infty} \quad (p \neq 1)$

This converges for $p > 1$ and diverges for $p \leq 1$.

For $p=1$, integral gives $\ln(\ln x)$ which also diverges

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges only for $p > 1$. L'Hôpital

Ratio test: $\frac{a_{n+1}}{a_n} = \frac{n}{n+1} \left[\frac{\ln n}{\ln(n+1)} \right]^p \rightarrow 1$, so inconclusive

Root test: $\sqrt[n]{a_n} = \frac{1}{\sqrt[n]{n} (\ln n)^{p/n}} \rightarrow 1$ $\left[\frac{1}{n} \ln n \rightarrow \frac{1}{n} \rightarrow 0 \Rightarrow \sqrt[n]{n} \rightarrow 1 \right]$
 $\frac{1}{n} \ln(\ln n) \rightarrow \frac{1}{n \ln n} \rightarrow 0 \Rightarrow \sqrt[n]{\ln n} \rightarrow 1$
 so inconclusive.

$$(3) (a) \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\ln n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{2} n^{-1/2}}{n^{-1}} = \lim_{n \rightarrow \infty} \frac{1}{2} \sqrt{n} \rightarrow \infty$$

(a), (b), (c)
each use
L'Hôpital

Since $a_n \not\rightarrow 0$, series diverges (nth term test)

$$(b) \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{-\frac{1}{n^2} \cos \frac{1}{n}}{-\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \cos \frac{1}{n} \rightarrow 1.$$

Since $a_n \not\rightarrow 0$, series diverges (nth term test)

$$(c) \lim_{n \rightarrow \infty} \frac{\tan^{-1} n}{1+n^2} = \lim_{n \rightarrow \infty} \frac{\frac{1}{1+n^2}}{2n} \ll \lim_{n \rightarrow \infty} \frac{1}{2n^3} \rightarrow 0, \text{ so may converge}$$

To examine whether it does converge, consider

$$\int_{\pi/4}^{\pi/2} \frac{8 \tan^{-1} x}{1+x^2} dx = \int_{y=\pi/4}^{y=\pi/2} \frac{8 y \sec^2 y}{1+\tan^2 y} dy \quad (\text{putting } x = \tan y)$$

$$= \int_{\pi/4}^{\pi/2} 8 y dy = 4 [y^2]_{\pi/4}^{\pi/2} = \frac{3\pi^2}{4} \Rightarrow \sum_{n=1}^{\infty} \frac{8 \tan^{-1} n}{1+n^2} \text{ converges by integral test}$$

$$(4) (a) \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} > \sum_{n=3}^{\infty} \frac{1}{\ln n} > \sum_{n=3}^{\infty} \frac{1}{n} \text{ since } \ln x < x \text{ [all terms positive if } n \geq 3]$$

Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{\ln(\ln n)}$ diverges (comparison test)

$$(b) \sum_{n=1}^{\infty} \frac{10n+1}{n(n+1)(n+2)} < \sum_{n=1}^{\infty} \frac{10n+1}{n^3} \approx \sum_{n=1}^{\infty} \frac{10}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^3}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ [Lectures], series converges.

$$(c) \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1 \text{ from 3(b)} \Rightarrow \sum_{n=1}^{\infty} \sin \frac{1}{n} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ converge or diverge together. Since } \sum \frac{1}{n} \text{ diverges, } \sum \sin \frac{1}{n} \text{ diverges.}$$

$$(5) (a) a_n = \frac{n\sqrt{2}}{2^n} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{2^n}{2^{n+1}} \left(\frac{n+1}{n} \right)^{\sqrt{2}} = \frac{1}{2} \left(1 + \frac{1}{n} \right)^{\sqrt{2}} \rightarrow \frac{1}{2}$$

Since limit is below 1, series converges (ratio test).

$$(b) a_n = \frac{(n!)^n}{(n^n)^2} \Rightarrow \sqrt[n]{a_n} = \frac{n!}{n^2} = \frac{n!}{n^2} = 1 \cdot \left(1 - \frac{1}{n} \right) (n-2)(n-3) \dots 1$$

Since this diverges as $n \rightarrow \infty$, series diverges (root test)

$$(c) a_n = \frac{1 \cdot 3 \dots (2n-1)}{4^n 2^n n!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(2n+1)}{4 \times 2 \times (n+1)} = \frac{2+\frac{1}{n}}{8(1+\frac{1}{n})} \rightarrow \frac{1}{4}$$

Since limit is below 1, series converges (ratio test)